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The modified Hamilton–Schwinger action principle

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Abstract. The c-number variation in the quantum-mechanical action principle of Schwinger is extended to a q-number variation which uses an action integral analogous to the classical modified Hamilton action integral. An explicit realization of the admissible q-number variation for the hamiltonian operator, H, given by

$$H(\boldsymbol{q}, \boldsymbol{p}, t) = \tfrac{1}{2}p_j g^{jk}(\boldsymbol{q})p_k + \tfrac{1}{2}\{A^j(\boldsymbol{q}), p_j\} + W(\boldsymbol{q}, t)$$

is discussed in terms of Gauteaux variation. The action principle yields the correct Hamilton–Heisenberg equations and a relationship between the lagrangian L and the hamiltonian H. It is also shown that while the fundamental commutation relation is successfully derived by Schwinger in his c-number variational principle, the same argument cannot be used in the present q-number variation. A new method of quantization is suggested.

1. Introduction

The need for a q-number variational principle in lagrangian formalism of quantum mechanics (QM), where the principle is applied to the action integral $\mathscr{I}$ defined by

$$\mathscr{I}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \int_{t'}^{t''} L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\,\mathrm{d}t, \qquad \dot{\boldsymbol{q}} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q} \tag{1.1}$$

has already been discussed by Cohen and Shaharir (1973a, b) in detail. In short, since the commutation relation (CR) between $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ is, in general, not a multiple of the identity operator (c number), the 'variation' in $\boldsymbol{q}$ (hence $\dot{\boldsymbol{q}}$) must be a q number. For example in the case where the CR

$$[\boldsymbol{q}, \dot{\boldsymbol{q}}] = \mathrm{i}\hbar \boldsymbol{g}(\boldsymbol{q}) \tag{1.2}$$

a c-number variation implies that $\boldsymbol{g}$ must be independent of the operator $\boldsymbol{q}$. Thus c-number variation is applicable only in the euclidean space.

We are interested in the possible extension of the so-called 'modified Hamilton variational principle' (Leech 1968), where the action integral (1.1) is (classically) substituted by a new functional $\mathscr{J}$ defined by

$$\mathscr{J}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{p}) = \int_{t'}^{t''} \left(\sum_{k=1}^{N} p_k \dot{q}^k - H(\boldsymbol{q}, \boldsymbol{p}, t)\right) \mathrm{d}t \tag{1.3a}$$

or in homogeneous formalism

$$\mathscr{J}(\boldsymbol{q}, t, \mathrm{d}\boldsymbol{q}, \mathrm{d}t, \boldsymbol{p}) = \int_{t'}^{t''} \left(\sum_{k=1}^{N} p_k\,\mathrm{d}q^k - H(\boldsymbol{q}, \boldsymbol{p}, t)\,\mathrm{d}t\right), \tag{1.3b}$$

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into the quantum domain. It is well known that the classical modified Hamilton homogeneously variational principle which is precisely prescribed by the Gauteaux variation

$$\begin{aligned}
&\delta\mathscr{J}(\boldsymbol{q}, t, \mathrm{d}\boldsymbol{q}, \mathrm{d}t, \boldsymbol{p}; \delta\boldsymbol{q}, \delta t, \delta(\mathrm{d}\boldsymbol{q}), \delta(\mathrm{d}t), \delta\boldsymbol{p})\\
&\quad\equiv \lim_{\epsilon\to 0}\epsilon^{-1}(\mathscr{J}(\boldsymbol{q}+\epsilon\delta\boldsymbol{q}, t+\epsilon\delta t, \mathrm{d}\boldsymbol{q}+\epsilon\delta(\mathrm{d}\boldsymbol{q}), \mathrm{d}t+\epsilon\delta(\mathrm{d}t), \boldsymbol{p}+\epsilon\delta\boldsymbol{p})-\mathscr{J}(\boldsymbol{q}, t, \mathrm{d}\boldsymbol{q}, \mathrm{d}t, \boldsymbol{p}))\\
&\quad= 0 \qquad (1.4)
\end{aligned}$$

such that

$$\delta\boldsymbol{q}(t') = \delta\boldsymbol{q}(t'') = \delta t(t') = \delta t(t'') = 0 \qquad (1.5)$$

$$\delta\boldsymbol{p}(t') = \delta\boldsymbol{p}(t'') = 0, \qquad (1.6)$$

$$\delta\mathrm{d} = \mathrm{d}\delta, \qquad (1.7)$$

leads to the Hamilton equations:

$$\dot{q}k = \frac{\partial H}{\partial p_k}(\boldsymbol{q}, \boldsymbol{p}, t), \qquad \frac{\mathrm{d}p_k}{\mathrm{d}t} = -\frac{\partial H}{\partial q_k}(\boldsymbol{q}, \boldsymbol{p}, t) \qquad (1.8)$$

$$\frac{\mathrm{d}H}{\mathrm{d}t}(\boldsymbol{q}, \boldsymbol{p}, t) = \frac{\partial H}{\partial t}(\boldsymbol{q}, \boldsymbol{p}, t). \qquad (1.9)$$

Similarly, the classical non-homogeneous variational principle yields equation (1.8) which is then used to derive (1.9) algebraically.

In quantum mechanics (QM), one tends to define an analogue of the action integral $\mathscr{J}$ in (1.3*a*) and (1.3*b*) as an operator $\mathscr{W}$ given respectively by

$$\mathscr{W}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{p}) = \int_{t'}^{t''}\left[\frac{1}{2}\sum_{k=1}^{N}\{p_k, \dot{q}^k\} - H(\boldsymbol{q}, \boldsymbol{p}, t)\right]\mathrm{d}t, \qquad (1.10a)$$

and

$$\mathscr{W}(\boldsymbol{q}, t, \mathrm{d}\boldsymbol{q}, \mathrm{d}t, \boldsymbol{p}) = \int_{t'}^{t''}\left(\frac{1}{2}\sum_{k=1}^{N}\{p_k, \mathrm{d}q^k\} - H(\boldsymbol{q}, \boldsymbol{p}, t)\,\mathrm{d}t\right), \qquad (1.10b)$$

where the variables $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, $\mathrm{d}\boldsymbol{q}$, $\boldsymbol{p}$ are hermitian operators but t and $\mathrm{d}t$ are c-number. ($\{A, B\}$ denotes anticommutator, $AB+BA$.) Then by postulating a quantum variation δ, in which $\delta\boldsymbol{q}$, $\delta\boldsymbol{p}$ and δt are multiples of identity (ie c numbers) and they satisfy conditions (1.4)–(1.7), the Hamilton–Heisenberg equations can be obtained in the usual manner. Similar results are obtained if one substitutes equations (1.4)–(1.6) with a single equation

$$\delta\mathscr{W} = J(t'') - J(t') \qquad (1.11)$$

for some operator J. This is essentially the 'action principle' of Schwinger (1953a, b, 1970), though he did not use the action integral (1.10*a*) or (1.10*b*) explicitly.

However it is clearly inconsistent to apply the same variational principle to the two invertible action integrals (1.1) and (1.10); the former has already been shown to be a q-number variation, whereas the latter is a c number. Furthermore since, as Schwinger (1957) has pointed out, the lagrangian L defined by the integrand (1.10*a*) is necessarily a euclidean model for the Boson system, there is *prima facie* evidence that his c-number action principle may not be correctly applied to a general lagrangian model in riemannian space. Moreover Lin *et al* (1970) had shown 'algebraically' that an acceptable lagrangian in a flat riemannian space is given by

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \tfrac{1}{2}\{p_k, \dot{q}^k\} + \frac{h^2}{4} g^{jk}\Gamma^m_{jn}\Gamma^n_{km} - H(\boldsymbol{q}, \boldsymbol{p}), \tag{1.12}$$

where g^{jk} is the contravariant metric tensor operator and Γ^l_{jk} is the usual Christoffel symbol (operators). It is clear, a c-number variational principle is not adequate, and Schwinger's action principle must be extended to a q-number variational principle.

In this paper we will postulate a q-number variational principle for the non-homogeneous and homogeneous action integral defined respectively as

$$\mathscr{S}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{p}) = \int_{t'}^{t''} \mathrm{d}t(\tfrac{1}{2}\{p_k, \dot{q}^k\} + \mathscr{D}(\boldsymbol{q}, \boldsymbol{p}, t) - H(\boldsymbol{q}, \boldsymbol{p}, t)) \tag{1.13a}$$

and

$$\mathscr{S}(\boldsymbol{q}, t, \mathrm{d}\boldsymbol{q}, \mathrm{d}t, \boldsymbol{p}) = \int_{t'}^{t''} [\tfrac{1}{2}\{p_k, \mathrm{d}q^k\} + (\mathscr{D}(\boldsymbol{q}, \boldsymbol{p}, t) - H(\boldsymbol{q}, \boldsymbol{p}, t))\,\mathrm{d}t]. \tag{1.13b}$$

The existence of the hermitian operator $\mathscr{D}$ will be postulated such that the q-number variation $\delta(\mathscr{D} - H)$ in $(\mathscr{D} - H)$ differs from

$$\frac{1}{2}\left\{\frac{\partial H}{\partial q^k}, \delta q^k\right\} + \frac{1}{2}\left\{\frac{\partial H}{\partial p_k}, \delta p_k\right\} + \frac{\partial H}{\partial t}\delta t$$

by at most a total derivative $\mathrm{d}\mathscr{G}/\mathrm{d}t$ of an operator $\mathscr{G}$. Following Schwinger we postulate further that the variation $\delta\mathscr{S}$ in $\mathscr{S}$ depends only on the end points t' and t'', namely

$$\delta\mathscr{S} = J(t'') - J(t'), \tag{1.14}$$

and the usual Hamilton–Heisenberg equations are obtained.

In § 3 an explicit realization of the q-number variational principle is discussed by considering an explicit lagrangian whose hamiltonian H is given by

$$H(\boldsymbol{q}, \boldsymbol{p}, t) = \tfrac{1}{2} p_j g^{jk}(\boldsymbol{q}) p_k + \tfrac{1}{2}\{p_j, A^j(\boldsymbol{q})\} + W(\boldsymbol{q}, t), \tag{1.15}$$

where $\det(g^{jk}) \neq 0$, and H is a hermitian operator. There the δ variation is defined in terms of Gauteaux variation as used in the classical 'calculus of variation' of Gel'fand and Fomin (1963) and Vainberg (1964). It is shown that the assumptions of the fundamental CR and the conditions derived from the admissible variation of the operator $[\boldsymbol{q}, \dot{\boldsymbol{q}}]$ yields the derived operator $\mathscr{D}$ and $\mathscr{G}$ postulated in § 2. It turns out that with the derived operator $\mathscr{D}$, the integrand in (1.13) coincides with equation (1.12). Thus it gives further partial explanation to the validity of the 'transformation' (1.12) in a general riemannian space.

In relation to the derivation of the fundamental CR via Schwinger's action principle, it is shown in § 4, that the same argument cannot be used for the present q-number variation. A new method of deriving the same CR is proposed.

2. The postulate of the *q*-number action principle

We define an operator $\mathscr{S}$, the quantum-modified Hamilton action integral as

$$\mathscr{S}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{p}) = \int_{t'}^{t''} \mathrm{d}t(\tfrac{1}{2}\{p_k, \dot{q}^k\} + \mathscr{D}(\boldsymbol{q}, \boldsymbol{p}, t) - H(\boldsymbol{q}, \boldsymbol{p}, t)) \tag{2.1}$$

or in homogeneous formulation

$$\mathscr{S}(\boldsymbol{q}, t, \mathrm{d}\boldsymbol{q}, \mathrm{d}t, \boldsymbol{p}) = \int_{t'}^{t''} [\tfrac{1}{2}\{p_k, \mathrm{d}q^k\} + (\mathscr{D}(\boldsymbol{q}, \boldsymbol{p}, t) - H(\boldsymbol{q}, \boldsymbol{p}, t))\,\mathrm{d}t]. \tag{2.2}$$

All operators are assumed to be hermitian. While the coordinate $\boldsymbol{q}$ and the momentum $\boldsymbol{p}$ are operators, the time variable t is assumed to be a c number. The operator $\mathscr{D}$, a function of $\boldsymbol{q}$ and $\boldsymbol{p}$ and t, is chosen such that it satisfies the following postulate of the q-number variational principle. There is a variational mapping δ, for which the variation $\delta\mathscr{D}$ in the operator $\mathscr{D}$ satisfies the identity

$$\delta\mathscr{D} - \frac{\mathrm{d}}{\mathrm{d}t}(\delta t\mathscr{D}) = \delta H - \frac{1}{2}\left\{\frac{\partial H}{\partial q^k}, \delta q^k\right\} - \frac{1}{2}\left\{\frac{\partial H}{\partial p_k}, \delta p_k\right\} - \frac{\partial H}{\partial t}\delta t. \tag{2.3}$$

Thus the operator $\mathscr{D}$ is determined uniquely by δH the variation in the hamiltonian H. In the case where t is not varied, we may assume a variation δ_t such that

$$\delta_t\mathscr{D} - \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{G} = \delta_t H - \frac{1}{2}\left\{\frac{\partial H}{\partial q^k}, \delta q^k\right\} - \frac{1}{2}\left\{\frac{\partial H}{\partial p_k}, \delta p_k\right\}. \tag{2.4}$$

Further, following Whittaker (1961) and Schwinger (1953a, b, 1970), we will assume that δ and $\mathrm{d}/\mathrm{d}t$ are commutative. In particular

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{q}) = \delta\left(\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t}\right), \qquad \frac{\mathrm{d}}{\mathrm{d}t}(\delta t) = \delta\left(\frac{\mathrm{d}t}{\mathrm{d}t}\right) = 0. \tag{2.5}$$

Following Schwinger again, we assume that the variation $\delta\mathscr{S}$ in the action integral (2.2) depends only on the end points t' and t'', ie

$$\delta\mathscr{S} = J(t'') - J(t') \tag{2.6}$$

for some operator J. A similar postulate may be made for the variation δ_t on the action integral (2.1) or (2.2).

Now assuming the usual properties of δ variation, we have by (2.2)

$$\begin{aligned}\delta\mathscr{S} = &\int_{t'}^{t''} [\tfrac{1}{2}\,\mathrm{d}(\{p_k, q^k\}) + \mathrm{d}((\mathscr{D} - H)\delta t)] \\ &+ \int_{t'}^{t''} (\tfrac{1}{2}\{\delta p_k, \mathrm{d}q^k\} - \tfrac{1}{2}\{\mathrm{d}p_k, \delta q^k\}) + \int_{t'}^{t''} [\delta(\mathscr{D} - H)\,\mathrm{d}t - \mathrm{d}(\mathscr{D} - H)\,\delta t],\end{aligned} \tag{2.7}$$

where use has also been made of equation (2.5). From equations (2.3) and (2.7) we derive

$$\begin{aligned}\delta\mathscr{S} = &\int_{t'}^{t''} \mathrm{d}(\tfrac{1}{2}\{p_k, \delta q^k\} + (\mathscr{D} - H)\,\delta t) + \int_{t'}^{t''} \frac{1}{2}\left\{\mathrm{d}q^k - \frac{\partial H}{\partial p_k}\mathrm{d}t, \delta p_k\right\} \\ &- \int_{t'}^{t''} \frac{1}{2}\left\{\mathrm{d}p_k + \frac{\partial H}{\partial q^k}\mathrm{d}t, \delta q^k\right\} + \int_{t'}^{t''} \left(\mathrm{d}H - \frac{\partial H}{\partial t}\mathrm{d}t\right)\delta t.\end{aligned} \tag{2.8}$$

Then assuming $\delta \boldsymbol{q}, \delta t, \delta \boldsymbol{p}$ are non-singular independent variables and applying the action principle (2.6), we may infer

$$\frac{\mathrm{d}}{\mathrm{d}t} q^k = \frac{\partial H}{\partial p_k}(\boldsymbol{q}, \boldsymbol{p}, t) \tag{2.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} p_k = \frac{-\partial H}{\partial q^k}(\boldsymbol{q}, \boldsymbol{p}, t) \tag{2.10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} H(\boldsymbol{q}, \boldsymbol{p}, t) = \frac{\partial H}{\partial t}(\boldsymbol{q}, \boldsymbol{p}, t) \tag{2.11}$$

and

$$J = \tfrac{1}{2}\{p_k, \delta q^k\} + (\mathscr{D} - H)\,\delta t. \tag{2.12}$$

Thus we have shown that the Hamilton–Heisenberg equations (2.9)–(2.11) can be derived from a variational principle which we will refer to as the modified Hamilton–Schwinger q-number homogeneous variational principle. Note that a similar calculation shows equations (2.9) and (2.10) can also be derived from the corresponding non-homogeneous variational principle, where the variational mapping δ_t is used instead of δ. However, unlike in the classical theory, equations (2.9) and (2.10) are not sufficient to be used to derive (2.11) algebraically, because $\boldsymbol{q}$ and $\boldsymbol{p}$ are non-commutative variables. For this reason the homogeneous formalism seems to be fundamentally more important than the non-homogeneous one. In fact the homogeneous formalism can be used to derive the CR between $\boldsymbol{q}$ and $\boldsymbol{p}$. Discussion on the possible derivation of this CR is deferred to § 4.

3. An explicit realization of the q-number variational principle

In this section we show an explicit realization of the q-number variation principle, which was postulated in the previous section, § 2, in terms of the Gauteaux variation. We will consider a class of operator H defined by

$$H(\boldsymbol{q}, \boldsymbol{p}, t) = \tfrac{1}{2}(p_j g^{jk}(\boldsymbol{q}) p_k + \{p_j, A^j(\boldsymbol{q})\}) + W(\boldsymbol{q}, t), \tag{3.1}$$

where $\boldsymbol{q}$ and $\boldsymbol{p}$ form a pair of canonical conjugate variables which satisfy the following commutation relations:

$$[q^j, q^k] = 0 = [p_l, p_m] \tag{3.2}$$

and

$$[q^j, p^k] = i h \delta^j_k. \tag{3.3}$$

For this purpose we may assume that the velocity operator $\dot{\boldsymbol{q}}$ is defined by

$$\dot{q}^k = \frac{\partial H}{\partial p_k}(\boldsymbol{q}, \boldsymbol{p}, t) = \tfrac{1}{2}(\{p_j, g^{jk}(\boldsymbol{q})\} + 2A^k(\boldsymbol{q})) \tag{3.4}$$

and hence $\dot{\boldsymbol{q}}$ satisfies the CR

$$[q^j, \dot{q}^k] = i h g^{jk}(\boldsymbol{q}). \tag{3.5}$$

It is sufficient to consider only the variation δ_t which keeps t unchanged.

Now by the definition of δ_t we have

$$\delta_t H = \tfrac{1}{2}(\delta p_j g^{jk} p_k + p_j g^{jk} \delta p_k) + \tfrac{1}{2}(\{p_j, \delta_t A^j\} + \{\delta p_j, A^j\}) + \tfrac{1}{2} p_j \, \delta g^{jk} p_k + \delta_t W. \tag{3.6}$$

Let us restrict ourselves to the admissible q-number variation $\delta \boldsymbol{q}$ of the operator $\boldsymbol{q}$ given by

$$\delta q^j = \alpha^j(\boldsymbol{q}, t; \lambda(t)) + \tfrac{1}{2}\beta(t)\{g^{jk}, p_k\} \tag{3.7}$$

where the c number $\lambda(t)$ and $\beta(t)$ are small independent parameters. This is a general admissible q-number variation for which the variation δO of the operator $O(\boldsymbol{q})$ function of $\boldsymbol{q}$ may be expressed as

$$\delta O = \frac{1}{2}\left\{\frac{\partial O}{\partial q^j}, \delta q^j\right\} \tag{3.8}$$

as discussed by Cohen and Shaharir (1973b). Thus equation (3.6) now may be written as

$$\begin{aligned}
\delta_t H = {} & \tfrac{1}{2}\{\delta p_j, \tfrac{1}{2}\{p_k, g^{jk}\} + A^j\} \\
& + \frac{1}{2}\left\{\frac{1}{2}\left\{p_j, \frac{\partial A^j}{\partial q_k}\right\}, \delta q^k\right\} + \frac{1}{2}\left\{\frac{1}{2} p_j \frac{\partial g^{jk}}{\partial q^l} p_k, \delta q^l\right\} + \frac{1}{2}\left\{\frac{\partial W}{\partial q^k}, \delta q^k\right\} \\
& + \tfrac{1}{4}[\delta p_j, [g^{jk}, p_k]] + \tfrac{1}{4} p_j \frac{\partial g^{jk}}{\partial q^l}[\delta q^l, p_k] - \tfrac{1}{4}[\delta q^l, p_k]\frac{\partial g^{jk}}{\partial q^l} p_j \\
& + \frac{1}{4}\left[[p_j, \delta q^k], \frac{\partial A^j}{\partial q^k}\right].
\end{aligned} \tag{3.9}$$

Define

$$\begin{aligned}
E = {} & [\delta p_j, [g^{jk}, p_k]] + \frac{1}{2}\left[p_j, \left\{\frac{\partial g^{jk}}{\partial q^l}, [\delta q^l, p_k]\right\}\right] + \frac{1}{2}\left\{p_k, \left[\frac{\partial g^{jk}}{\partial q^l}, [\delta q^l, p_j]\right]\right\} \\
& + \left[[p_j, \delta q^k], \frac{\partial A^j}{\partial q^k}\right]
\end{aligned} \tag{3.10}$$

Equation (3.9) can now be written as

$$\delta_t H = \frac{1}{2}\left\{\frac{\partial H}{\partial p_k}, \delta p_k\right\} + \frac{1}{2}\left\{\frac{\partial H}{\partial q^k}, \delta q^k\right\} + \frac{1}{4}E. \tag{3.11}$$

Now, by definition, we have

$$\delta[p_j, O(\boldsymbol{q})] = [\delta p_j, O(\boldsymbol{q})] + \frac{1}{2}\left[p_j, \left\{\frac{\partial O(\boldsymbol{q})}{\partial q^k}, \delta q^k\right\}\right]. \tag{3.12}$$

Since

$$[p_j, O(\boldsymbol{q})] = -\mathrm{i}h \frac{\partial O(\boldsymbol{q})}{\partial q^j} \tag{3.13}$$

hence from (3.12), we deduce

$$[\delta p_j, O] = -\frac{1}{2}\left\{\frac{\partial O}{\partial q^k}, [p_j, \delta q^k]\right\}. \tag{3.14}$$

Further using equations (3.13) and (3.7), the CR between $\delta \boldsymbol{q}$ and $\boldsymbol{p}$ may be settled. Thus we obtain

$$[\delta q^j, p_k] = \mathrm{i}h\frac{\partial \alpha^j}{\partial q^k} + \frac{\mathrm{i}h}{2}\beta(t)\left\{\frac{\partial g^{jl}}{\partial q^k}, p_l\right\}, \tag{3.15}$$

and deduce

$$[p_l, [\delta q^j, p_k]] = h^2\frac{\partial^2 \alpha^j}{\partial q^k \partial q^l} + \frac{h^2}{2}\beta(t)\left\{\frac{\partial^2 g^{jm}}{\partial q^k \partial q^l}, p_m\right\}. \tag{3.16}$$

Applying the results in (3.13)–(3.16) into (3.10) we deduce

$$E = h^2\left(\frac{\partial^2 \alpha^j}{\partial q^k \partial q^l} - \beta(t)\frac{\partial^2 A^j}{\partial q^k \partial q^l}\right)\frac{\partial g^{kl}}{\partial q^j}. \tag{3.17}$$

Using similar arguments as those in (3.12)–(3.14), the existence and uniqueness of the variation $\delta([q^j, \dot{q}^k])$ in the commutator q^j with $\dot{q}^k$, (3.5), implies

$$g^{jl}(\alpha^k - \beta(t)A^k)_{;l} + g^{kl}(\alpha^j - \beta(t)A^j)_{;l} + g^{jk}\frac{\mathrm{d}\beta}{\mathrm{d}t} = 0, \tag{3.18}$$

where we have defined

$$v^j_{;k} = \frac{\partial v^j}{\partial q^k} + \Gamma^j_{kl}v^l \tag{3.19}$$

the contravariant derivative of a vector component v^j. It is also easy to show that, for any continuously twice differentiable vector field $\boldsymbol{v}$ (Cohen and Shaharir 1973b)

$$\frac{\partial g^{jk}}{\partial q^l}\frac{\partial^2 v^l}{\partial q^k \partial q^j} = \frac{\partial}{\partial q^l}(g^{jk}\Gamma^m_{jn}\Gamma^n_{km})v^l + 2g^{kl}\Gamma^m_{jn}\Gamma^n_{km}v^j_{;l} - 2g^{kl}\Gamma^m_{jn}\Gamma^n_{km}v^k_{;ml}. \tag{3.20}$$

By applying equations (3.18) and (3.20) into equation (3.17) we obtain

$$E = h^2\frac{\partial}{\partial q^l}(g^{jk}\Gamma^m_{jn}\Gamma^n_{km})(\alpha^l - \beta(t)A^l). \tag{3.21}$$

Equation (3.21) may further be simplified to give

$$E = \frac{h^2}{2}\left\{\frac{\partial}{\partial q^l}(g^{jk}\Gamma^m_{jn}\Gamma^n_{km}), \delta q^l\right\} - h^2\frac{\mathrm{d}}{\mathrm{d}t}(\beta(t)\Gamma^m_{jn}\Gamma^n_{km}g^{jk}). \tag{3.22}$$

By (3.11) and (3.22) we may identify the operators $\mathscr{D}$ and $\mathscr{G}$, so that equations (2.3) and (2.4) are satisfied. The action integral may now be defined explicitly:

$$\mathscr{S}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{p}) = \int_{t'}^{t''}\mathrm{d}t(\tfrac{1}{2}\{p_k, \dot{q}^k\} + \tfrac{1}{4}h^2\Gamma^m_{jn}\Gamma^n_{km}g^{jk} - H(\boldsymbol{q}, \boldsymbol{p}, t)) \tag{3.23}$$

and the action principle described in § 2 yields the correct canonical equation:

$$\dot{p}_k = -\frac{\partial H}{\partial q^k}$$

(provided $\delta \boldsymbol{q}$ is non-singular).

The above analysis suffices to show that the q-number variational principle postulated in § 2 is not empty. The result also gives further partial explanation to the justification of adopting the 'transformation'

$$L = \tfrac{1}{2}\{p_k, \dot{q}^k\} + \tfrac{1}{4}\hbar^2 g^{jk}\Gamma^m_{jn}\Gamma^n_{km} - H(\boldsymbol{q}, \boldsymbol{p}, t) \tag{3.24}$$

without the additional term $\frac{1}{4}\hbar^2 R$ which otherwise may arise from the algebraic method of Lin *et al* (1970) and Kimura (1971). (R is the usual Riemann curvature scalar.)

Indeed, earlier Lin *et al* had determined $\mathscr{D}$ in flat riemannian space by insisting that the expression $\frac{1}{2}\{p_k, \dot{q}^k\} + \mathscr{D}$ becomes $\frac{1}{2}\{M_k, \dot{X}^k\}$ and vice versa under the point transformation

$$\boldsymbol{q} \Leftrightarrow \boldsymbol{X}: X^k = X^k(\boldsymbol{q}), \qquad q^k = q^k(\boldsymbol{X})$$

$$\boldsymbol{p} \Leftrightarrow \boldsymbol{M}: M_k = \frac{1}{2}\left\{p_l, \frac{\partial q^l}{\partial X^k}\right\}, \qquad p_k = \frac{1}{2}\left\{M_l, \frac{\partial X^l}{\partial q^k}\right\}, \tag{3.25}$$

where $\boldsymbol{X}$ and $\boldsymbol{M}$ are coordinate and momentum in an euclidean space respectively. The flatness of the space ensures that the transformation (3.25) exists everywhere in the space. A tedious calculation shows that under the above transformation

$$\{M_k, \dot{X}^k\} = \{p_k, \dot{q}^k\} + \tfrac{1}{2}\hbar^2 R + \tfrac{1}{2}\hbar^2 g^{jk}\Gamma^m_{jn}\Gamma^n_{km}, \tag{3.26}$$

as found by Lin *et al.* Since the space is flat (ie $R = 0$) the relation between L and H in the flat riemannian space is exactly given by (3.24). Clearly this algebraic method of determining $\mathscr{D}$ in a general riemannian space is ambiguous. Locally, it looks as though the term involving R should be retained if the space is curved. However later, Kimura (1971) insisted that the relation (3.24) must also be true in all riemannian space by incorporating the term $\frac{1}{4}\hbar^2 R$ into the lagrangian or hamiltonian. Our result through the above action principle shows that Kimura's argument is somewhat dubious.

4. Fundamental commutation relation

Schwinger (1953a, b, 1970), in his c-number 'action principle' had obtained the fundamental CR (3.2) and (3.3) by identifying the operators

$$T_{\boldsymbol{p}} = -\delta p_k q^k, \qquad T_{\boldsymbol{q}} = p_k \delta q^k \tag{4.1}$$

induced by the 'action principle', as the infinitesimal generators for the momentum translation and spatial translation respectively. Essentially the identification of the generators in (4.1) is possible only if the Heisenberg equation

$$\frac{\mathrm{d}O}{\mathrm{d}t} - \frac{\partial O}{\partial t} = \frac{1}{\mathrm{i}\hbar}[O, H] \tag{4.2}$$

and the Hamilton–Heisenberg equations (2.9)–(2.11), yield the CR

$$[T_{\boldsymbol{p}}, H]\frac{1}{\mathrm{i}\hbar} = \frac{\partial H}{\partial p_k}\delta p_k \tag{4.3}$$

$$[H, T_{\boldsymbol{q}}]\frac{1}{\mathrm{i}\hbar} = \frac{\partial H}{\partial q^k}\delta q^k. \tag{4.4}$$

However in the present q-number variation the above argument can no longer be used. Indeed, considering the simplest q-number variation where $\delta \boldsymbol{q}$ and $\delta \boldsymbol{p}$ are functions of $\boldsymbol{q}$ and $\boldsymbol{p}$ respectively, the variational principle provides us with generators

$$J_{\boldsymbol{q}} = \tfrac{1}{2}\{p_k, \delta q^k\} \tag{4.5}$$

$$J_{\beta} = -\tfrac{1}{2}\{q^k, \delta p_k\}. \tag{4.6}$$

Then Schwinger's argument fails to give equations (4.7) and (4.8):

$$\frac{1}{\mathrm{i}\hbar}[q^k, J_{\boldsymbol{q}}] = \delta q^k \tag{4.7}$$

$$\frac{1}{\mathrm{i}\hbar}[p_k, J_{\boldsymbol{p}}] = \delta p_k. \tag{4.8}$$

Thus, while in c-number variation equations (4.7) and (4.8) may be derived, but for the q-number variation, the same equations need to be postulated. These results are not surprising, since the CR obtained through the c-number variational principle are just sufficient conditions for the consistency between the canonical equations (2.9)–(2.11) and the essentially postulated Heisenberg equations for the operators $\boldsymbol{q}$ and $\boldsymbol{p}$. The following, we propose a simple derivation of the CR without using the Heisenberg equation, (4.2).

We have already seen that for a class of hamiltonian given by equation (3.1), the fundamental CR (3.2) and (3.3) are sufficient to yield a consistent formulation of the q-number variational principle. Now let us define

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \lim_{\delta t \to 0} \frac{H(\boldsymbol{q}(t+\delta t), \boldsymbol{p}(t+\delta t), t+\delta t) - H(\boldsymbol{q}, \boldsymbol{p}, t)}{\delta t} \tag{4.9}$$

then since δt is a c number, the right-hand side of equation (4.1) may be reduced to

$$\lim_{\delta t \to 0} \frac{H(\boldsymbol{q}+\delta t\dot{\boldsymbol{q}}, \boldsymbol{p}+\delta t\dot{\boldsymbol{p}}, t) - H(\boldsymbol{q}, \boldsymbol{p}, t)}{\delta t} + \frac{\partial H(\boldsymbol{q}, \boldsymbol{p}, t)}{\partial t}.$$

However the variational principle requires $\mathrm{d}H/\mathrm{d}t$ equal to the partial derivative $\partial H/\partial t$ of H with respect to time. Thus we must have a condition

$$\lim_{\delta t \to 0} \frac{H(q+\delta t\dot{\boldsymbol{q}}, \boldsymbol{p}+\delta t\dot{\boldsymbol{p}}, t) - H(\boldsymbol{q}, \boldsymbol{p}, t)}{\delta t} = 0, \tag{4.10}$$

where $\dot{\boldsymbol{q}}$ and $\dot{\boldsymbol{p}}$ satisfy equations (2.9) and (2.10) respectively. It can be shown that the CR listed in (3.2) and (3.3) are sufficient for the validity of the equation (4.10).

We conclude that while the generator J induced by the c-number variational principle may naturally serve to obtain the fundamental CR it is not so in the case of the q-number variational principle.

There remains a possibility of deriving the CR via the (q-number) variational principle of Peierls (1952) which we will discuss further in a later work.

5. Conclusion

We have extended the modified Schwinger's c-number 'action principle' (1953a, b, 1970) to the admissible q-number variational principle. However, unlike the c-number variational principle where the transformational generator induced by the 'action principle' may naturally serve to determine the fundamental commutatives relation, it is no longer true in the present formulation. A new simple method of deriving the fundamental commutation relation is suggested. This is done by drawing a sufficient condition for the consistency between the derived Hamilton–Heisenberg equations and the intrinsic definition of the total derivative of the hamiltonian.

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